NORTH SYDNEY BOYS HIGH SCHOOL

2022 Mathematics Extension 2

Assessment Task 3

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using blue or black pen
- Calculators approved by NESA may be used
- A separate reference sheet is provided
- For questions in Section 2, show relevant mathematical reasoning and/or calculations
- Begin each question in a new writing booklet
- Write your student number on each writing booklet

Teacher

- Mr Berry
- Ms Lee
- Mr Umakanthan

Total Marks: 100

Section 1 – 10 marks (pages 3 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section 2 – 90 marks (pages 7 – 12)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Student Number: ……………………………………………………………………

Multiple Choice	Question 11	Question 12	Question 13	Question 14	Question 15	Question 16	Total %
$\overline{10}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{100}$

Section 1

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions $1 - 10$

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1 What is i^{2022}?

A. 1

B. i

C. -1

D. $-i$

Questions 2 and 3 both refer to the following statement and selection of answers.

Statement

"If I go to *Efficiency Coaching* I will become efficient."

Answers

A. If I do not go to *Efficiency Coaching* I will not become efficient.

B. If I did not become efficient I did not go to *Efficiency Coaching*.

C. I went to *Efficiency Coaching* and I did not become efficient.

D. I did not go to *Efficiency Coaching* and I became efficient.

2 Which of the above is the negation of the statement?

3 Which of the above is the contrapositive of the statement?

4 Which of the following describes a set of points, $z \in \mathbb{C}$, that lie on a circle with a radius of 5 and a centre of $(0,0)$?

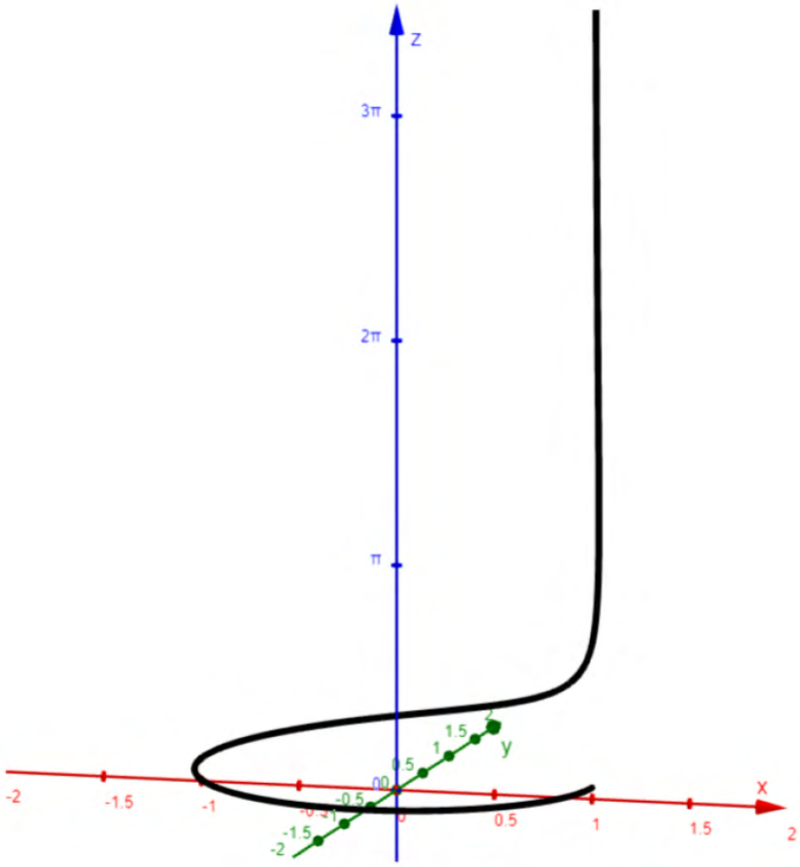
A. $z\bar{z} = 5$

B. $z^2 = 25$

C. $Re(z^2) + Im(z^2) = 25$

D. $(z + \bar{z})^2 - (z - \bar{z})^2 = 100$

5



Which of the following parametric equations best describes the graph above.

A. $x = \cos t\,,\ y = \sin t,\ z = t$

B. $x = \cos t\,,\ y = \sin t,\ z = \frac{1}{t}$

C. $x = \cos t\ , y = t,\ z = \sin t$

D. $x = \cos t\ , y = \frac{1}{t},\ z = \sin t$

6 Evaluate:

$$\int_{-a}^{a} \frac{e^x}{e^x + e^{-x}}\,dx$$

A. 0

B. a

C. $\tan^{-1}(e^a) - \tan^{-1}(e^{-a})$

D. $\ln(e^{2a} - 1) - a$

7 The position vector of the centre of a sphere is $\underset{\sim}{c} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ and the sphere is defined such that $|\underset{\sim}{v} - \underset{\sim}{c}| = 3$, where $\underset{\sim}{v}$ gives the position vectors of the points on the surface of the sphere. Which of the following is the cartesian equation of a sphere which is tangent to $\underset{\sim}{v}$?

A. $(x-5)^2 + (y-7)^2 + (z+2)^2 = 36$

B. $(x-3)^2 + (y-3)^2 + (z-2)^2 = 9$

C. $(x-4)^2 + (y-2)^2 + (z-8)^2 = 18$

D. $x^2 + y^2 + z^2 = 21$

8 If z lies on the circle $|z| = 2$, find the minimum value of:

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right|$$

A. $\frac{1}{35}$

B. $\frac{1}{19}$

C. $\frac{1}{3}$

D. $\frac{1}{2}$

9 A particle is initially stationary at position $x = 1$ metres. It experiences an acceleration of $\ddot{x} = 2x - 4\ \text{m/s}^2$.

What is the speed of the particle after it has travelled for a total of 5 metres?

A. $2\sqrt{2}\ \text{m/s}$

B. $\sqrt{30}\ \text{m/s}$

C. $\sqrt{70}\ \text{m/s}$

D. $4\sqrt{6}\ \text{m/s}$

10 Which of the following is a possible solution to $\sin x = 2$?

A. $4i$

B. $\frac{i\sqrt{2}}{1+\sqrt{2}} - \pi$

C. $\pi - e^{i\sqrt{2}}$

D. $\frac{\pi}{2} + i\ln(2+\sqrt{3})$

Section 2

90 Marks
Attempt Questions $11-16$
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

For questions in Section 2, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks)

(a) $z = 1 + 2i$ is a solution to the cubic equation $2z^3 - z^2 + 4z + k = 0$, where k is a real number.

(i) Find the other two solutions. **2**

(ii) Evaluate k. **1**

(b) Points A, B and C are given by the position vectors $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ respectively.

Find:

(i) $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$ **2**

(ii) $\angle ABC$ **2**

(iii) The ***<u>exact</u>*** area of ΔABC **2**

(c) Shade the region on the Argand diagram that is satisfied by both the inequalities

$|z - 3 + i| \leq 3$ and $|z| \geq |z - 2i|$ **3**

(d) Prove that $\sqrt{3k+2}$ is not an integer for all positive integers k. **3**

Question 12 (15 Marks)

(a) The displacement of a particle is given by $x = \sqrt{3}\sin 2t - \cos 2t + 3$.

(i) Find the initial displacement and velocity of the particle. **2**

(ii) Prove that the particle is moving in simple harmonic motion. **1**

(iii) Find v^2 in terms of x. **2**

(iv) Hence or otherwise find the maximum displacement and maximum speed of the particle. **2**

(b) Find the point on the line $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ that is closest to the origin. **3**

(c) On a standard Argand diagram, the complex number $\sqrt{3} + i$ represents one of the vertices of a regular hexagon, with centre at the origin O.

(i) Find the complex numbers which represent the other 5 vertices **2**

(ii) The complex numbers that represent the vertices are all raised to the power of 4, creating a closed shape *S*, whose sides are straight line segments.

Find the area of *S*. **3**

Question 13 (15 Marks)

(a) By considering the partial fraction

$$\frac{1}{x^4 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x^2 + 1}$$

find:

$$\int \frac{1}{x^4 - 1}\,dx$$ **3**

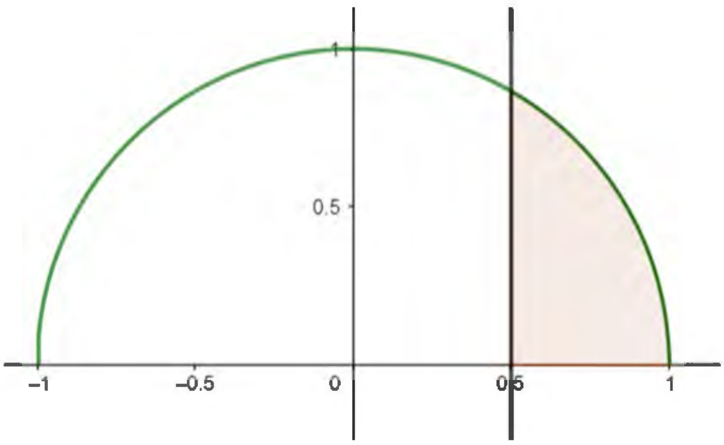
(b) Find

$$\int x^2 \sin x\,dx$$ **3**

(c) Use an appropriate substitution to evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{4}{3 + 5\cos x}\,dx$$ **3**

(d) The diagram below shows $y = \sqrt{1 - x^2}$ and $x = \frac{1}{2}$



Find the exact shaded area enclosed between $y = \sqrt{1 - x^2}$, $x = \frac{1}{2}$ and the *x*-axis.

3

(e) Given that $a = 3^{-\frac{1}{6}}$ and $b = 3^{\frac{1}{6}}$, using the substitution $u = \frac{1}{x}$ or otherwise, evaluate:

$$\int_a^b \frac{x^3}{(1 + x)(1 + x^6)}\,dx$$ **3**

Question 14 (15 Marks)

(a) (i) $f(z) = z^6 + 8z^3 + 64 \quad , \quad z \in \mathbb{C}$

Given that $f(z) = 0$, show that

$z^3 = -4 \pm 4\sqrt{3}i$ **1**

(ii) Hence or otherwise find all six solutions to the equation $f(z) = 0$, giving answers in the form $z = re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$. **2**

(iii) Hence show that:

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{6\pi}{9} + \cos\frac{8\pi}{9} = -\frac{1}{2}$$ **3**

(b) Mr Berry confiscates an *Efficiency Coaching* booklet from a student (who should not have been doing their tutoring homework in class) and drops it from the window of B14.

The booklet lands on the playground exactly 1.5 seconds later.

The booklet weighs 0.2 kg and experiences a force due to air resistance of magnitude $0.4v$ Newtons, where v is its velocity.

Letting gravity be $9.8\ \mathrm{m/s^2}$, find to two decimal places:

(i) The speed of the booklet when it hits the ground **3**

(ii) The percentage of its terminal velocity it reached at the moment of impact **1**

(iii) The height the booklet was dropped from **3**

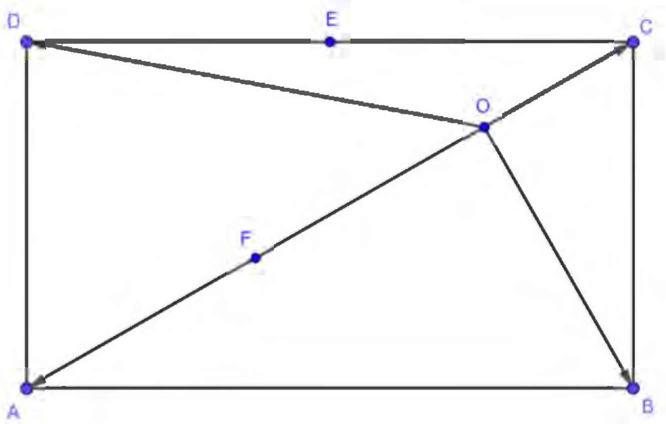
(c) Given that $z \in \mathbb{C}$ such that $Im(z) \neq 0$ and $\frac{z}{1+z^2}$ is a real number. Show that $|z| = 1$ **2**

Question 15 (15 Marks)

(a) In the diagram below, O is the origin and points A, B, C and D have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$ respectively.

E is the midpoint of CD and F is the midpoint of OA.

$ABCD$ is a rectangle such that O lies on diagonal AC, and OB is perpendicular to AC



(i) Show that $|\underset{\sim}{b}|^2 + \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$ **2**

(ii) Hence, or otherwise, show that $\angle BFE = \frac{\pi}{2}$ **3**

(b) (i) Expand and simplify $3[(x-2)^3 + 2]$ **1**

(ii) Given

$$I_n = \int_0^a \frac{x^n}{\sqrt{a^2 - x^2}}\,dx\,,\ n \in \mathbb{N}$$

clearly show that:

$$I_n = \frac{a^2(n-1)}{n} I_{n-2}\ \ ,\ n \geq 2$$

3

(iii) Show that

$$\int_2^4 \frac{3x^3 - 18x^2 + 36x - 18}{\sqrt{4x - x^2}}\,dx = 3I_3 + 3\pi$$

4

(iv) Hence or otherwise, evaluate

$$\int_2^4 \frac{3x^3 - 18x^2 + 36x - 18}{\sqrt{4x - x^2}}\,dx$$

2

Question 16 (15 Marks)

(a) (i) By considering the expansion of $k^3 - (k-1)^3$ show that:

$$\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1)$$ **2**

(do ***NOT*** *use mathematical induction)*

(ii) Prove that there are no finite sets of prime numbers where $2^2 + 3^2 + \cdots + n^2$ is a multiple of at least one element of the set for all integers $n \geq 2$. **2**

(b) (i) Let $\underset{\sim}{w} = \underset{\sim}{u} + r\underset{\sim}{v}$ where $\underset{\sim}{w}$, $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are vectors and r is a scalar.

Prove that $|\underset{\sim}{v}|^2 r^2 + 2(\underset{\sim}{u} \cdot \underset{\sim}{v})r + |\underset{\sim}{u}|^2 \geq 0$ **1**

(ii) Use your result from (i) to show that $\underset{\sim}{u} \cdot \underset{\sim}{v} \leq |\underset{\sim}{u}||\underset{\sim}{v}|$ **2**

(iii) Use your result from part (ii) to show that for real numbers x, y and z.

$$xy + yz + zx \leq x^2 + y^2 + z^2$$ **1**

(iv) Use your result from part (iii) to show that for non-negative values of x, y and z:

$$xyz \leq \frac{x^3 + y^3 + z^3}{3}$$ **1**

(v) Given that $a + b + c + d = 0$ and $a^2 + b^2 + c^2 + d^2 = 12$. Find:

1. The maximum value of $abcd$ **2**

2. The minimum value of $abcd$ **4**

1 What is i^{2022}?

A. 1

B. i

C. -1

D. $-i$

$i^{2022} = i^{2020} \times i^2$

$= ((i)^4)^{505} \times -1$

$= 1^{505} \times -1$

$= 1 \times -1$

$= -1$

Questions 2 and 3 both refer to the following statement and selection of answers.

Statement

"If I go to *Efficiency Coaching* I will become efficient."

Answers

A. If I do not go to *Efficiency Coaching* I will not become efficient.

B. If I did not become efficient I did not go to *Efficiency Coaching*.

C. I went to *Efficiency Coaching* and I did not become efficient.

D. I did not go to *Efficiency Coaching* and I became efficient.

2 Which of the above is the negation of the statement? C

3 Which of the above is the contrapositive of the statement? B

4 Which of the following describes a set of points, $z \in \mathbb{C}$, that lie on a circle with a radius of 5 and a centre of $(0,0)$?

A. $z\bar{z} = 5$ ← $r = \sqrt{5}$

B. $z^2 = 25$ ← not a circle

C. $Re(z^2) + Im(z^2) = 25$ ← not a circle

D. $(z + \bar{z})^2 - (z - \bar{z})^2 = 100$

$z = x + iy$

$z + \bar{z} = 2x$

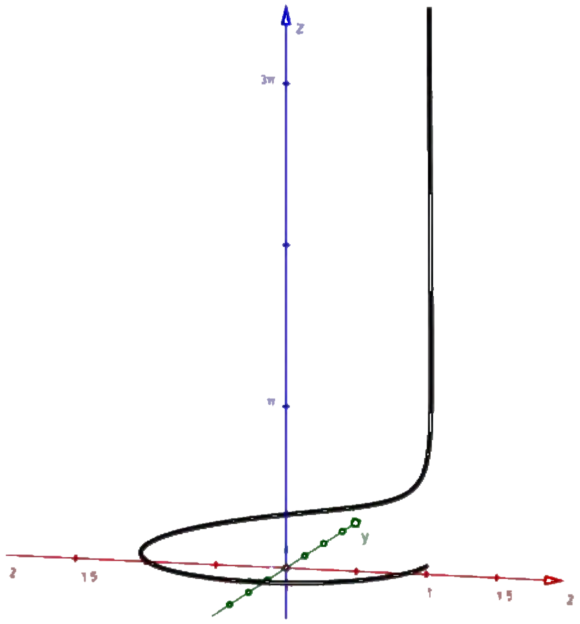
$z - \bar{z} = 2iy$

$(2x)^2 - (2iy)^2$

$4x^2 + 4y^2 = 100$

$x^2 + y^2 = 25$

5



at $t = 0$ we see
$x = 1$, $y = 0$ &
z is undefined

Which of the following parametric equations best describes the graph above.

A. $x = \cos t\,,\; y = \sin t,\; z = t$

B. $x = \cos t\,,\; y = \sin t,\; z = \frac{1}{t}$

C. $x = \cos t\;, y = t,\; z = \sin t$

D. $x = \cos t\;, y = \frac{1}{t},\; z = \sin t$

6 Evaluate:

$$\int_{-a}^{a} \frac{e^x}{e^x + e^{-x}} dx$$

A. 0

B. a

C. $\tan^{-1}(e^a) - \tan^{-1}(e^{-a})$

D. $\ln(e^{2a} - 1) - a$

$$\int_{-a}^{a} \frac{e^x}{e^x + e^{-x}}\, dx$$

$$= \int_0^a \frac{e^x}{e^x + e^{-x}}\, dx + \int_0^a \frac{e^{-x}}{e^{-x} + e^x}\, dx$$

$$= \int_0^a \frac{e^x + e^{-x}}{e^x + e^{-x}}\, dx$$

$$= \int_0^a 1\, dx$$

$$= [x]_0^a$$

$$= a$$

7 The position vector of the centre of a sphere is $\underset{\sim}{c} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ and the sphere is defined such that $|\underset{\sim}{v} - \underset{\sim}{c}| = 3$, where $\underset{\sim}{v}$ gives the position vectors of the points on the surface of the sphere. Which of the following is the cartesian equation of a sphere which is tangent to $\underset{\sim}{v}$?

radius

A. $(x-5)^2 + (y-7)^2 + (z+2)^2 = 36$ ← radius = 6

B. $(x-3)^2 + (y-3)^2 + (z-2)^2 = 9$

sum of radii = 6+3
= 9

C. $(x-4)^2 + (y-2)^2 + (z-8)^2 = 18$

D. $x^2 + y^2 + z^2 = 21$

distance of centres

$d^2 = (2-5)^2 + (1-7)^2 + (4--2)^2$
$= 3^2 + 6^2 + 6^2$
$= 81$
$d = 9$

distance between centres = sum of radii

8 If z lies on the circle $|z| = 2$, find the minimum value of:

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right|$$

$$\frac{1}{|z^4 - 4z^2 + 3|}$$

$$= \frac{1}{|z^2 - 3|\,|z^2 - 1|}$$

A. $\frac{1}{35}$

B. $\frac{1}{19}$

C. $\frac{1}{3}$

$$\geq \frac{1}{(|z^2| + |-3|)(|z^2| + |-1|)}$$

D. $\frac{1}{2}$

triangle inequality

$$= \frac{1}{(4+3)(4+1)}$$

$$= \frac{1}{7 \times 5}$$

$$= \frac{1}{35}$$

9 A particle is initially stationary at position $x = 1$ metres. It experiences an acceleration of $\ddot{x} = 2x - 4\ \mathrm{m/s^2}$.

What is the speed of the particle after it has travelled for a total of 5 metres?

A. $2\sqrt{2}\ \mathrm{m/s}$

B. $\sqrt{30}\ \mathrm{m/s}$

C. $\sqrt{70}\ \mathrm{m/s}$

D. $4\sqrt{6}\ \mathrm{m/s}$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x - 4$$

$$\frac{d}{dx}v^2 = 4x - 8$$

$$\left[v^2\right]_0^v = \left[2x^2 - 8x\right]_1^x$$

$$v^2 = 2x^2 - 8x - (2-8)$$

$$v^2 = 2x^2 - 8x + 6$$

$$= 2(x^2 - 4x + 3)$$

$$= 2(x-1)(x-3)$$

$$\therefore v = \pm\sqrt{2(x-1)(x-3)}$$

$$v = \pm\sqrt{2\times(-4-1)(-4-3)}$$

$$= \pm\sqrt{70}$$

$$\therefore \text{speed} = \sqrt{70}\ \mathrm{m/s}$$

as initial velocity is 0 and acceleration is negative particle always moves left so $x = -4$ when it has moved 5 metres

10 Which of the following is a possible solution to $\sin x = 2$?

A. $4i$

B. $\frac{i\sqrt{2}}{1+\sqrt{2}} - \pi$

C. $\pi - e^{i\sqrt{2}}$

D. $\frac{\pi}{2} + i\ln(2+\sqrt{3})$

$$\frac{e^{ix} - e^{-ix}}{2i} = 2$$

$$e^{ix} - \frac{1}{e^{ix}} = 4i$$

$$(e^{ix})^2 - 1 = (4i)e^{ix}$$

$$(e^{ix})^2 - 4i(e^{ix}) - 1 = 0$$

$$e^{ix} = \frac{4i \pm \sqrt{(4i)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{4i \pm \sqrt{-16+4}}{2}$$

$$= \frac{4i \pm \sqrt{-12}}{2}$$

$$= \frac{4i \pm 2\sqrt{3}\,i}{2}$$

$$e^{ix} = (2 \pm \sqrt{3})i$$

$$ix = \log_e[(2 \pm \sqrt{3})i]$$

$$= \log_e i + \log_e(2 \pm \sqrt{3})$$

$$= \log_e e^{i\frac{\pi}{2}} + \log_e(2 \pm \sqrt{3})$$

$$ix = i\left(\frac{\pi}{2}\right) + \log_e(2 \pm \sqrt{3})$$

$$x = \pi/2 + \frac{\log_e(2 \pm \sqrt{3})}{i} \times \frac{i}{i}$$

$$= \frac{\pi}{2} - i\log_e(2 \pm \sqrt{3})$$

$$= \frac{\pi}{2} - i\log_e(2 - \sqrt{3})$$ one of the solutions

$$= \frac{\pi}{2} - i\log_e\left(\frac{1}{2+\sqrt{3}}\right)$$

$$= \frac{\pi}{2} - i\log_e(2+\sqrt{3})^{-1}$$

$$= \frac{\pi}{2} + i\log_e(2+\sqrt{3})$$

Question 11 (15 Marks)

(a) $z = 1 + 2i$ is a solution to the cubic equation $2z^3 - z^2 + 4z + k = 0$, where k is a real number.

(i) Find the other two solutions. **2**

$\alpha = 1+2i$
$\beta = 1-2i$ (conjugate root theorem)

$\alpha + \beta + \gamma = -\left(-\frac{1}{2}\right)$
$\therefore 1 + \cancel{2i} + 1 - \cancel{2i} + \gamma = \frac{1}{2}$
$2 + \gamma = \frac{1}{2}$
$\gamma = -\frac{3}{2}$

(ii) Evaluate k. **1**

$\alpha\beta\gamma = -\frac{k}{2}$

$(1+2i)(1-2i)\left(\cancel{-}\frac{3}{\cancel{2}}\right) = \cancel{-}\frac{k}{\cancel{2}}$

$\therefore k = 3(1+2i)(1-2i)$
$= 3(1^2+2^2)$
$= 3 \times 5$
$= 15$

(b) Points A, B and C are given by the position vectors $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ respectively.

Find:

(i) $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$ **2**

$$|\overrightarrow{AB}|^2 = (1-2)^2 + (4-5)^2 + (-2-1)^2$$
$$= (-1)^2 + (-1)^2 + (-3)^2$$
$$= 11$$
$$\therefore |\overrightarrow{AB}| = \sqrt{11}$$

$$|\overrightarrow{BC}|^2 = (2-(-1))^2 + (5-1)^2 + (1-4)^2$$
$$= 3^2 + 4^2 + 3^2$$
$$= 34$$
$$\therefore |\overrightarrow{BC}| = \sqrt{34}$$

(ii) $\angle ABC$ **2**

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}| \cos\theta$$

$$\therefore \cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{AB}||\overrightarrow{BC}|}$$

product

$$= \frac{(-1)(-3) + (-1)(-4) + (-3)(3)}{\sqrt{11} \times \sqrt{34}}$$

$$= \frac{-2}{\sqrt{374}}$$

$$\therefore \angle ABC = \cos^{-1}\left(\frac{-2}{\sqrt{374}}\right)$$

answer

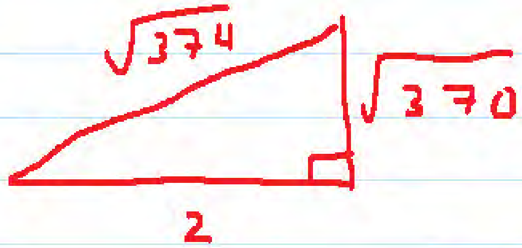
$$\overrightarrow{BA} = \begin{pmatrix} -1 \\ -1 \\ -3 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} -3 \\ -4 \\ 3 \end{pmatrix}$$

(iii) The ***exact*** area of ΔABC **2**

$$\text{Area} = \frac{1}{2}\,|\overrightarrow{AB}|\,|\overrightarrow{BC}|\,\sin\angle ABC$$

$$= \frac{1}{2} \times \sqrt{11} \times \sqrt{34} \times \sin\left[\cos^{-1}\left(\frac{2}{\sqrt{374}}\right)\right]$$



(not to scale)

$$= \frac{1}{2} \times \cancel{\sqrt{374}} \times \frac{\sqrt{370}}{\cancel{\sqrt{374}}}$$

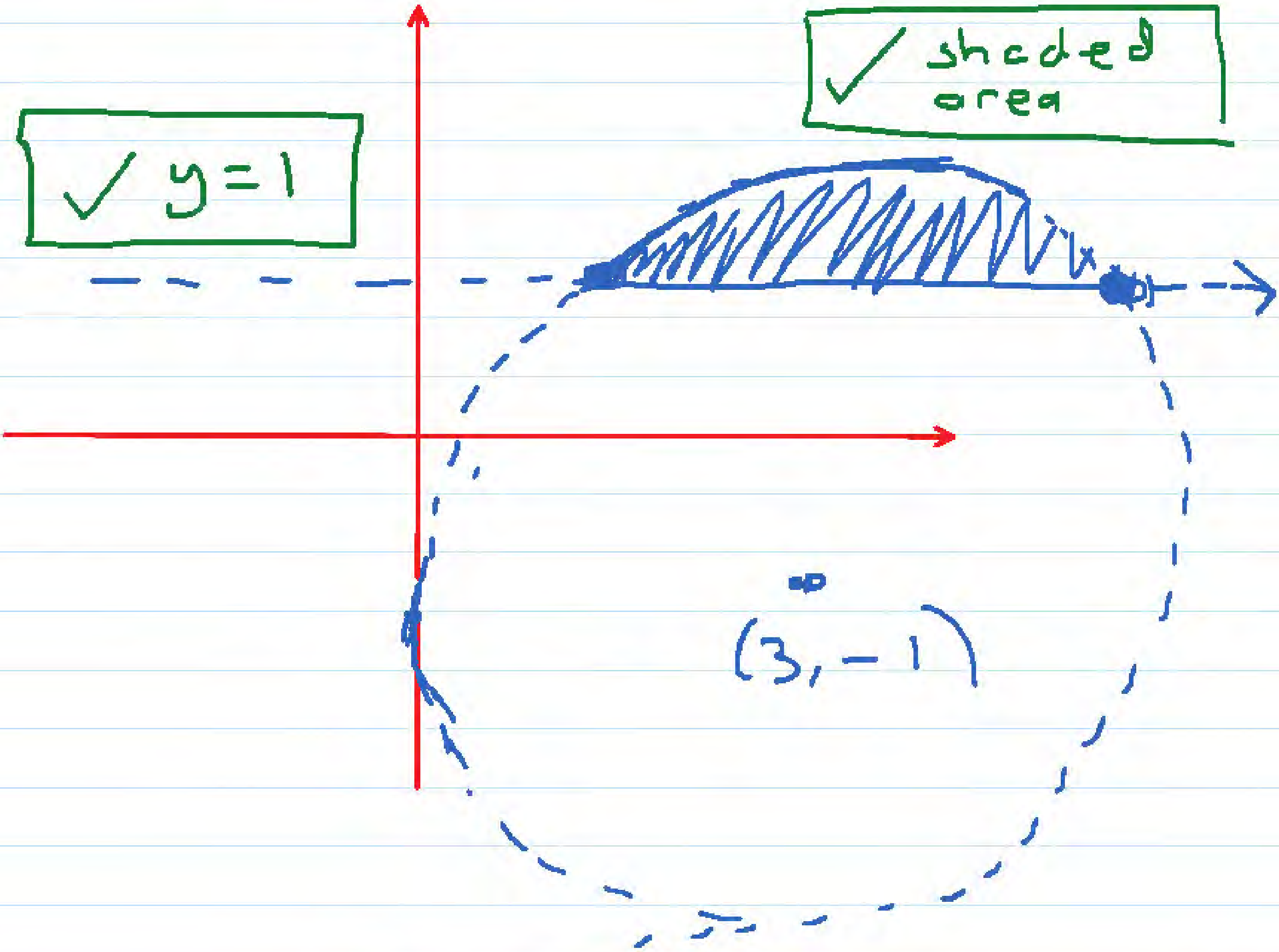
$$= \frac{\sqrt{370}}{2} \text{ units}^2$$

(c) Shade the region on the Argand diagram that is satisfied by both the inequalities

$|z - 3 + i| \leq 3$ and $|z| \geq |z - 2i|$ **3**

$|z - (3 - i)| \leq 3$ is inside a circle with centre $(3, -1)$ & radius 3

$|z| \geq |z - 2i|$ is above the perpendicular bisector of $y = 0$ & $y = 2$.



✓ $(x-3)^2 + (y+1)^2 = 3$

(d) Prove that $\sqrt{3k+2}$ is not an integer for all positive integers k. **3**

Assume integer P exists such that

$$p = \sqrt{3k+2}$$
$$\therefore p^2 = 3k+2$$

$\boxed{\checkmark \text{ assumption}}$

case 1 $p = 3n$ $(n \in \mathbb{Z})$

$$\therefore (3n)^2 = 3k+2$$
$$9n^2 = 3k+2$$
$$3(3n^2) = 3k+2$$

(impossible as LHS is a multiple of 3 while RHS is not)

$\boxed{\checkmark \text{ at least 1 case}}$

case 2 $p = 3n+1$ $(n \in \mathbb{Z})$

$$(3n+1)^2 = 3k+2$$
$$9n^2 + 6n + 1 = 3k+2$$
$$3(3n^2 + 2n) + 1 = 3k+2$$

(impossible as LHS is one more than a multiple of 3 while RHS is 2 more)

case 3 $p = 3n+2$ $(n \in \mathbb{Z})$

$$(3n+2)^2 = 3k+2$$
$$9n^2 + 12n + 4 = 3k+2$$
$$3(3n^2 + 4n + 1) + 1 = 3k+2$$

(impossible since LHS is one more than a multiple of 3 while RHS is 2 more)

$\therefore$ there are no possible values for p where $p^2 = 3k+2$

proof by contradiction

$\boxed{\checkmark \text{ all cases and conclusion}}$

Question 12 (15 Marks)

(a) The displacement of a particle is given by $x = \sqrt{3}\sin 2t - \cos 2t + 3$.

(i) Find the initial displacement and velocity of the particle. **2**

when $t = 0$

$x = \sqrt{3}\sin(2(0)) - \cos(2(0)) + 3$

$= 0 - 1 + 3$

$= 2\text{m}$ ✓

$\dot{x} = 2\sqrt{3}\cos 2t + 2\sin 2t$

when $t = 0$

$\dot{x} = 2\sqrt{3}\cos(2(0)) + \cancel{2\sin(2(0))}$

$= 2\sqrt{3}\ \text{m/s}$ ✓

(ii) Prove that the particle is moving in simple harmonic motion. **1**

$\ddot{x} = -4\sqrt{3}\sin 2t + 4\cos 2t$

$= -4(\sqrt{3}\sin 2t - \cos 2t)$

$= -4(x-3)$ ✓

$\therefore$ particle is moving in simple harmonic motion as acceleration is proportional to displacement in the opposide direction

(iii) Find v^2 in terms of x. **2**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4(x-3)$$

$$\frac{d}{dx}v^2 = -8(x-3)$$

$$\left[v^2\right]_{2\sqrt{3}}^{v} = \int_2^x -8x+24\,dx$$

✓ integration

$$v^2-(2\sqrt{3})^2 = \left[-4x^2+24x\right]_2^x$$

$$v^2-12 = -4x^2+24x-(-4\times 2^2+24(2))$$

$$= -4x^2+24x-32$$

$$\therefore v^2 = -4x^2+24x-20$$

✓ using initial conditions to get answer

(iv) Hence or otherwise find the maximum displacement and maximum speed of the particle. **2**

$$v^2 = -4(x^2-6x+5)$$

$$= -4(x-5)(x-1)$$

$\therefore v^2 = 0$ at $x=5$ & $x=1$

$\therefore$ max displacement at $x=5$ ✓

v^2 is maximised at $\frac{5+1}{2} = 3$

$$\therefore v^2 = -4(3-5)(3-1)$$

$$= 16$$

$\therefore$ max speed $= 4$ m/s ✓

(b) Find the point on the line $\underset{\sim}{r} = \begin{pmatrix}1\\4\\6\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix}$ that is closest to the origin. **3**

$$d^2 = (1+2\lambda)^2 + (4+\lambda)^2 + (6+\lambda)^2$$

✓ distance formula

$$= 1 + 4\lambda + 4\lambda^2 + 16 + 8\lambda + \lambda^2 + 36 + 12\lambda + \lambda^2$$

$$= 6\lambda^2 + 24\lambda + 53$$

find λ to minimise d^2 (and hence d)

✓ minimise distance

$$-\frac{24}{2(6)} = -\frac{24}{12}$$

$$= -2$$

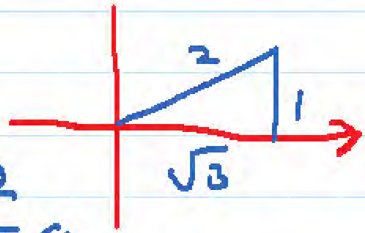
$\therefore$ point is $\begin{pmatrix}1\\4\\6\end{pmatrix} - 2\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}-3\\2\\4\end{pmatrix}$

✓ answer

(c) On a standard Argand diagram, the complex number $\sqrt{3} + i$ represents one of the vertices of a regular hexagon, with centre at the origin O.

(i) Find the complex numbers which represent the other 5 vertices **2**

$\sqrt{3} + i$



modulus $= 2$

argument $\pi/6$

$\sqrt{3} + 1 = 2e^{\pi/6\, i}$

✓ euler form

✓ vertices

$\therefore$ vertices are

$2e^{\pi/2\, i}$, $2e^{5\pi/6\, i}$, $2e^{-\pi/6\, i}$, $2e^{-\pi/2\, i}$, $2e^{-5\pi/6\, i}$

(found by adding $\frac{2\pi}{6} = \frac{\pi}{3}$ to the argument)

(ii) The complex numbers that represent the vertices are all raised to the power of 4, creating a closed shape S, whose sides are straight line segments.

Find the area of S. **3**

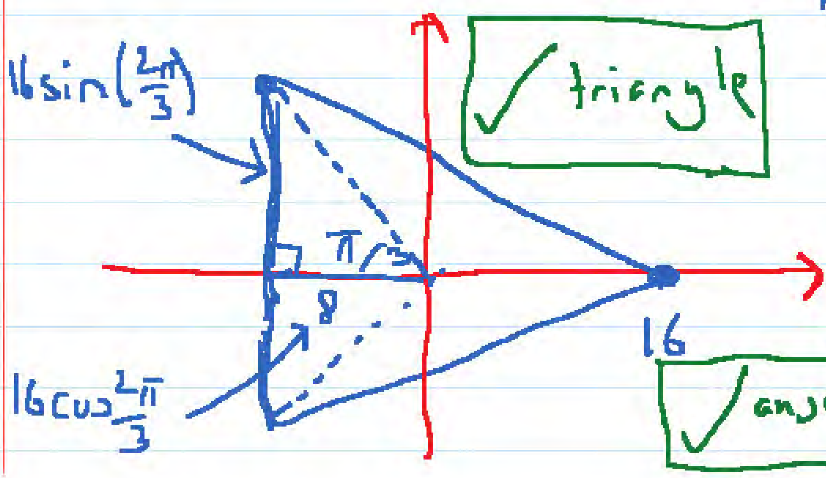
✓ values

raising answers in (i) to 4 we get

$16e^{2\pi i}$, $16e^{\frac{10}{3}\pi i}$, $16e^{-\frac{2}{3}\pi i}$, $16e^{-2\pi i}$, $16e^{-\frac{10}{3}\pi i}$, $16e^{\frac{2}{3}\pi i}$

the original $2e^{\pi/6}$

noting the duplicates, 3 unique values create a triangle



✓ triangle

height $= 2 \times 16\sin\left(\pi/3\right)$
$= 16\sqrt{3}$

base $= 16 + 16 \times \cos\frac{\pi}{3}$
$= 24$

$A = \frac{1}{2} \times 16\sqrt{3} \times 24$

✓ answer $= 192\sqrt{3}$ units^2

Question 13 (15 Marks)

(a) By considering the partial fraction

$$\frac{1}{x^4 - 1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x^2+1}$$

find:

$$\int \frac{1}{x^4 - 1}\,dx$$

3

$A(x+1)(x^2+1) + B(x-1)(x^2+1) + C(x^2-1) = 1$

$x = 1$

$A(2)(2) + 0 + 0 = 1$

$A = 1/4$

$x = -1$

$0 + B(-2)(2) = 1$

$B = -1/4$

✓ A, B & C

$x = 0$

$\frac{1}{4}(1)(1) - \frac{1}{4}(-1)(1) + C(-1) = 1$

$\frac{1}{2} - C = 1$

$C = -\frac{1}{2}$

✓ at least one correct integration

$$\int \frac{1}{x^4-1}\,dx = \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\tan^{-1}(x) + C$$

$$= \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}(x) + C$$

✓ answer

(b) Find

$$\int x^2 \sin x \, dx$$

3

$$\int \underset{u}{x^2} \underset{v'}{\sin x} \, dx = \underset{v}{(-\cos x)} \underset{u}{x^2} - \int \underset{u'}{2x} \underset{v}{(-\cos x)} \, dx$$

✓ 1st parts

$$= -x^2 \cos x + 2 \int \underset{u}{x} \underset{v'}{\cos x} \, dx$$

$$= -x^2 \cos x + 2 \left[\underset{u}{x} \underset{v}{\sin x} + \int \underset{u'v}{\sin x} \, dx \right]$$

✓ 2nd parts

$$= -x^2 \cos x + 2x \sin x - 2 \cos x + C$$

✓ answer

(c) Use an appropriate substitution to evaluate: **3**

$$\int_0^{\frac{\pi}{2}} \frac{4}{3+5\cos x}\,dx$$

sub into integral ✓

$$\int_0^{\frac{\pi}{2}} \frac{4}{3+5\cos x}\,dx$$

$$= \int_0^1 \frac{4}{3+5\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{8}{3+3t^2+5-5t^2}\,dt$$

$$= \int_0^1 \frac{8}{8-2t^2}\,dt$$

$$= \int_0^1 \frac{\cancel{2}\times 4}{\cancel{2}(4-t^2)}\,dt$$

$$= \int_0^1 \frac{4}{4-t^2}\,dt$$

let $t = \tan\frac{x}{2}$

$\therefore \frac{x}{2} = \tan^{-1} t$

$x = 2\tan^{-1} t$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$dx = \frac{2}{1+t^2}\,dt$

✓ sub limits

when $x = \pi/2$

$t = \tan \pi/4$

$= 1$

when $x = 0$

$t = \tan 0$

$= 0$

$$= \int_0^1 \left(\frac{1}{2+t} + \frac{1}{2-t}\right)dt$$

$$= \Big[\ln|2+t| - \ln|2-t|\Big]_0^1$$

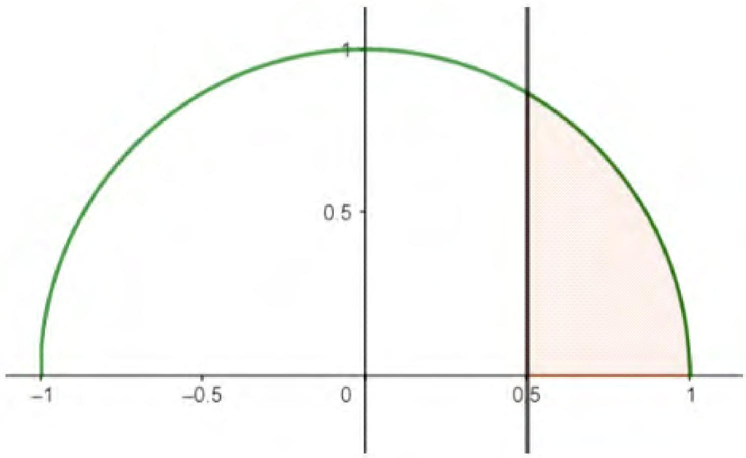
$$= \left[\ln\left|\frac{2+t}{2-t}\right|\right]_0^1$$

$$= \ln 3 - \cancel{\ln\left(\frac{2}{2}\right)}$$

$= \ln 3$

✓ answer

(d) The diagram below shows $y = \sqrt{1-x^2}$ and $x = \frac{1}{2}$



Find the exact shaded area enclosed between $y = \sqrt{1-x^2}$, $x = \frac{1}{2}$ and the x-axis. **3**

$$A = \int_{0.5}^{1} \sqrt{1-x^2}\, dx$$

$$= \int_{\pi/6}^{\pi/2} \underbrace{\sqrt{1-\sin^2\theta}}_{\cos\theta}\, \cos\theta\, d\theta$$

✓ sub into integral

$$= \int_{\pi/6}^{\pi/2} \cos^2\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_{\pi/6}^{\pi/2}$$

$$= \left(\frac{\pi/2}{2} + \frac{1}{4}\sin\left(\cancel{2\times\pi/2}\right)\right) - \left(\frac{\pi/6}{2} + \frac{1}{4}\sin\left(2\times\frac{\pi}{6}\right)\right)$$

$$= \pi/4 - \pi/12 - 1/4\left(\sqrt{3}/2\right)$$

$x = \sin\theta$

✓ limits

$\frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta\, d\theta$

$\theta = \sin^{-1} x$

when $x = 1$

$\theta = \frac{\pi}{2}$

when $x = 0.5$

$\theta = \frac{\pi}{6}$

$$= \left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) \text{units}^2 \quad \boxed{\checkmark \text{ answer}}$$

(e) Given that $a = 3^{-\frac{1}{6}}$ and $b = 3^{\frac{1}{6}}$, using the substitution $u = \frac{1}{x}$ or otherwise, evaluate:

$$\int_a^b \frac{x^3}{(1+x)(1+x^6)}\,dx$$

3

$$\int_{3^{-1/6}}^{3^{1/6}} \frac{x^3}{(1+x)(1+x^6)}\,dx$$

$u = \frac{1}{x}$

$\frac{du}{dx} = -\frac{1}{x^2}$

$dx = -du\,x^2$

$= -\frac{du}{u^2}$

when $x = 3^{1/6}$, $u = 3^{-1/6}$

when $x = 3^{-1/6}$, $u = 3^{1/6}$

swap limits

$$= \int_{3^{1/6}}^{3^{-1/6}} \frac{(1/u)^3}{(1+1/u)(1+\frac{1}{u^6})}\left(-\frac{du}{u^2}\right)$$

✓ substitution

$$= \int_{3^{-1/6}}^{3^{1/6}} \frac{1}{u^5(1+1/u)(1+1/u^6)}\,du$$

u u^4

$$= \int_{3^{-1/6}}^{3^{1/6}} \frac{1}{(u+1)(u^4+\frac{1}{u^2})}\,du \times \frac{u^2}{u^2}$$

$$= \int_{3^{-1/6}}^{3^{1/6}} \frac{u^2}{(1+u)(1+u^6)}\,du \qquad \boxed{u = x}$$

$$\therefore \int_{3^{-1/6}}^{3^{1/6}} \frac{x^3}{(1+x)(1+x^6)}\,dx = \int_{3^{-1/6}}^{3^{1/6}} \frac{x^2}{(1+x)(1+x^6)}\,dx$$

$$\therefore 2\int_{3^{-1/6}}^{3^{1/6}} \frac{x^3}{(1+x)(1+x^6)}\,dx = \int_{3^{-1/6}}^{3^{1/6}} \frac{x^3+x^2}{(1+x)(1+x^6)}\,dx$$

✓ addition of integrals

$$\int_{3^{-1/6}}^{3^{1/6}} \frac{x^3}{(1+x)(1+x^6)}\,dx = \frac{1}{2}\int_{3^{-1/6}}^{3^{1/6}} \frac{x^2(\cancel{1+x})}{(\cancel{1+x})(1+x^6)}\,dx$$

$$= \frac{1}{2}\int_{3^{-1/6}}^{3^{1/6}} \frac{x^2}{1+(x^3)^2}\,dx$$

$$= \frac{1}{6}\int_{3^{-1/6}}^{3^{1/6}} \frac{3x^2}{1+(x^3)^2}\,dx$$

$$= \frac{1}{6}\left[\tan^{-1}(x^3)\right]_{3^{-1/6}}^{3^{1/6}}$$

$$= \frac{1}{6}\left[\tan\sqrt{3} - \tan\left(\frac{1}{\sqrt{3}}\right)\right]$$

$3^{\frac{1}{2}}$ $\quad$ $3^{-\frac{1}{2}}$

$$= \frac{1}{6}\left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$$= \frac{1}{6} \times \frac{\pi}{6}$$

$$= \frac{\pi}{36}$$

$\boxed{\checkmark \text{ answer}}$

Question 14 (15 Marks)

(a) (i) $f(z) = z^6 + 8z^3 + 64 = 0 \quad , \quad z \in \mathbb{C}$

Given that $f(z) = 0$, show that

$z^3 = -4 \pm 4\sqrt{3}i$ **1**

$$1(z^3)^2 + 8(z^3) + 64 = 0$$

$a \qquad b \qquad c$

$$z^3 = \frac{-8 \pm \sqrt{8^2 - 4(1)(64)}}{2(1)}$$

✓ quadratic formula

$$= \frac{-8 \pm \sqrt{-192}}{2}$$

$$= \frac{-8 \pm 8\sqrt{-3}}{2}$$

$$= -4 \pm 4\sqrt{3}\,i$$

(ii) Hence or otherwise find all six solutions to the equation $f(z) = 0$, giving answers in the form $z = re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$. **2**

$r^3 e^{3\theta i} = 8e^{2\pi/3\, i}$ for $z^3 = -4 + 4\sqrt{3}\,i$

$r = 2 \qquad 3\theta = 2\pi/3,\ 8\pi/3,\ -4\pi/3$

$\theta = 2\pi/9,\ 8\pi/9,\ -4\pi/9$

$r^3 = 8e^{-\frac{2\pi}{3}i}$ for $z^3 = -4 - 4\sqrt{3}\,i$

$r = 2 \qquad 3\theta = -\frac{2\pi}{3},\ -\frac{8\pi}{3},\ \frac{4\pi}{3}$

$\theta = -2\pi/9,\ -8\pi/9,\ 4\pi/9$

$$\therefore z = 2e^{\pm 2\pi/9\, i},\ 2e^{\pm 4\pi/9\, i},\ 2e^{\pm 8\pi/9\, i}$$

✓ any solution

✓ all solutions

(iii) Hence show that:

$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{6\pi}{9}+\cos\frac{8\pi}{9}=-\frac{1}{2}$$ **3**

we know the roots of $z^6+8z^3+64=0$ are.

$2e^{2\pi/9 i},\ 2e^{-2\pi/9 i},\ 2e^{4\pi/9 i},\ 2e^{-4\pi/9 i},\ 2e^{8\pi/9 i}\ \&\ 2e^{-8\pi/9 i}$

sum of roots $= -0/1$

$\boxed{\checkmark \text{ sum of roots}}$

$\therefore 2\left[\underbrace{e^{2\pi/9 i}+e^{-2\pi/9 i}}+\underbrace{e^{4\pi/9 i}+e^{-4\pi/9 i}}+\underbrace{e^{8\pi/9 i}+e^{-8\pi/9 i}}\right]=0$

$\therefore 2\left[2\cos{}^{2\pi}/_9+2\cos{}^{4\pi}/_9+2\cos{}^{8\pi}/_9\right]=0$

$\boxed{\checkmark\ e^{i\theta}+e^{-i\theta}=2\cos\theta}$

$\therefore \cancel{4}\left[\cos{}^{2\pi}/_9+\cos{}^{4\pi}/_9+\cos{}^{8\pi}/_9\right]=0$

$\therefore \cos{}^{2\pi}/_9+\cos{}^{4\pi}/_9+\underline{\cos{}^{6\pi}/_9}+\cos{}^{8\pi}/_9=\underline{\cos{}^{6\pi}/_9}$

added to each side

$\therefore \cos{}^{2\pi}/_9+\cos{}^{4\pi}/_9+\cos{}^{6\pi}/_9+\cos{}^{8\pi}/_9=-1/2$

$\boxed{\checkmark}$ answer / exact value of $\cos{}^{6\pi}/_9$

(b) Mr Berry confiscates an *Efficiency Coaching* booklet from a student (who should not have been doing their tutoring homework in class) and drops it from the window of B14.

The booklet lands on the playground exactly 1.5 seconds later.

The booklet weighs 0.2 kg and experiences a force due to air resistance of magnitude $0.4v$ Newtons, where v is its velocity.

Letting gravity be $9.8\ \mathrm{m/s^2}$, find to two decimal places:

(i) The speed of the booklet when it hits the ground **3**

$F = ma$

$\uparrow 0.4v$

$\downarrow 0.2g$

let down be positive

$0.2a = 0.2g - 0.4v$

$a = g - 2v$

$\dfrac{dv}{dt} = g - 2v$

✓ equation

$\dfrac{dt}{dv} = \dfrac{1}{g - 2v}$

$\displaystyle\int_0^t dt = -\frac{1}{2}\int_0^v \frac{-2}{g-2v}$

✓ integral

$t - 0 = -\frac{1}{2}\left[\ln|g - 2v| - \ln|g|\right]$

$t = -\frac{1}{2}\ln\left|\dfrac{g-2v}{g}\right|$

$-2t = \ln\left|\dfrac{g-2v}{g}\right|$

$\dfrac{g-2v}{g} = e^{-2t}$

$g - 2v = ge^{-2t}$

$2v = g - ge^{-2t}$

$v = g/2 - \dfrac{ge^{-2t}}{2}$

✓ answer

when $t = 1.5$

$v = 4.66\ m/s$

(2. d.p)

(ii) The percentage of its terminal velocity it reached at the moment of impact **1**

terminal velocity $= 4.9$ m/s

$$\therefore \frac{4.656\ldots}{4.9} \times 100\% = 95.02\%$$

(2 d.p.)

✓

(iii) The height the booklet was dropped from **3**

$$\frac{dx}{dt} = \frac{g}{2} - \frac{ge^{-2t}}{2}$$

$$\int_0^x dx = \frac{1}{2}\int_0^{1.5} (g - ge^{-2t})\, dt$$

✓ set up integral

$$x - 0 = \frac{1}{2}\left[gt + \frac{g}{2}e^{-2t}\right]_0^{1.5}$$

✓ primitive

$$x = \frac{1}{2}\left[9.8(1.5) + \frac{9.8}{2}e^{-2(1.5)} - \left(0 + \frac{9.8}{2}(1)\right)\right]$$

$$x = 5.02\text{m}$$

(2 d.p.)

✓ answer

(c) Given that $z \in \mathbb{C}$ such that $Im(z) \neq 0$ and $\frac{z}{1+z^2}$ is a real number. Show that $|z| = 1$ **2**

as $\frac{z}{1+z^2}$ is real the $\frac{1+z^2}{z}$ is also real

$$\frac{1+z^2}{z} = z + \frac{1}{z}$$

$$= x + iy + \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$

$$= x + iy + \frac{x-iy}{x^2+y^2}$$

$$= x + \frac{x}{x^2+y^2} + \left(y - \frac{y}{x^2+y^2}\right)i$$

$$\therefore y - \frac{y}{x^2+y^2} = 0$$

$\boxed{\checkmark \text{ using } Im(z) = 0}$

$$y\left(1 - \frac{1}{x^2+y^2}\right) = 0$$

as $y \neq 0$

$$1 - \frac{1}{x^2+y^2} = 0$$

$$1 = \frac{1}{x^2+y^2}$$

$$x^2+y^2 = 1$$

$$\therefore |z| = 1$$

$\boxed{\checkmark \text{ answer}}$

(c) Given that $z \in \mathbb{C}$ such that $Im(z) \neq 0$ and $\frac{z}{1+z^2}$ is a real number. Show that $|z| = 1$

Alternative solution

$\frac{z}{1+z^2}$ is real

$$\therefore \frac{z}{1+z^2} = \overline{\frac{z}{1+z^2}}$$

$$= \frac{\bar{z}}{\overline{1+z^2}}$$

✓ using conjugates

$$\frac{z}{1+z^2} = \frac{\bar{z}}{1+\bar{z}^2}$$

$$\therefore z + \underbrace{z\bar{z}}\,\bar{z} = \bar{z} + \underbrace{\bar{z}z}\,z$$

$$z + |z|^2\bar{z} = \bar{z} + |z|^2 z$$

$$z - \bar{z} + |z|^2\bar{z} - |z|^2 z = 0$$

$$z - \bar{z} + |z|^2(\bar{z} - z) = 0$$

$$z - \bar{z} - |z|^2(z - \bar{z}) = 0$$

$$(z - \bar{z})(1 - |z|^2) = 0$$

$\therefore |z|^2 = 1$ as $z \neq \bar{z}$ since $Im(z) \neq 0$

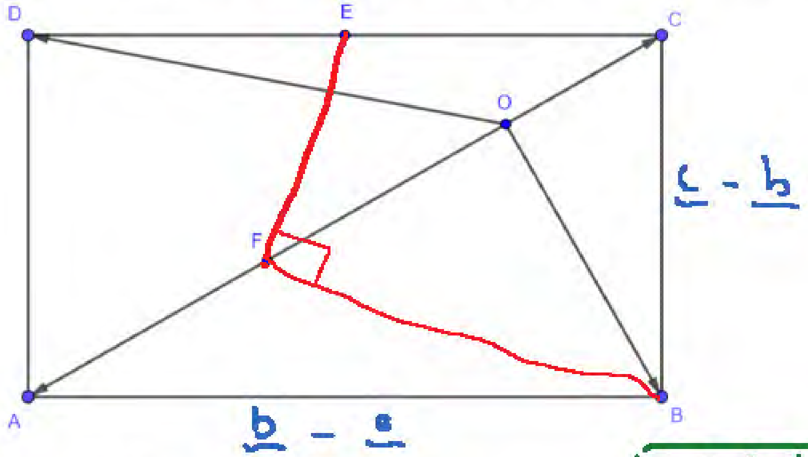
$\therefore |z| = 1$ $(as\ |z| > 0)$ ✓ answer

Question 15 (15 Marks)

(a) In the diagram below, O is the origin and points A, B, C and D have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$ respectively.

E is the midpoint of CD and F is the midpoint of OA.

$ABCD$ is a rectangle such that O lies on diagonal AC, and OB is perpendicular to AC



(i) Show that $|\underset{\sim}{b}|^2 + \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$ **2**

✓ dot product

$\vec{AB} \cdot \vec{BC} = 0$ (sides of rectangle perpendicular)

$(\underset{\sim}{b} - \underset{\sim}{a})(\underset{\sim}{c} - \underset{\sim}{b}) = 0$

$\cancel{\underset{\sim}{b}\underset{\sim}{c}} - \underset{\sim}{b}\,\underset{\sim}{b} - \underset{\sim}{a}\,\underset{\sim}{c} + \cancel{\underset{\sim}{a}\underset{\sim}{b}} = 0$

$\therefore -|\underset{\sim}{b}|^2 - \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$ (as $\underset{\sim}{a} \perp \underset{\sim}{b}$ & $\underset{\sim}{b} \perp \underset{\sim}{c}$)

$(\times -1)$ $\therefore |\underset{\sim}{b}|^2 + \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$ ✓ answer

(ii) Hence, or otherwise, show that $\angle BFE = \frac{\pi}{2}$ **3**

$\vec{BF} = \vec{BO} + \vec{OF}$
$= -\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}$
$= \frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b}$

✓ $\vec{BF}$, $\vec{FE}$

$\vec{FE} = \vec{FO} + \vec{OC} + \vec{CE}$
$= -\frac{1}{2}\underset{\sim}{a} + \underset{\sim}{c} + \frac{1}{2}\underbrace{(\underset{\sim}{a} - \underset{\sim}{b})}_{\vec{DC} = \vec{AB}}$

$= -\cancel{\tfrac{1}{2}}\underset{\sim}{a} + \underset{\sim}{c} + \cancel{\tfrac{1}{2}}\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}$
$= \underset{\sim}{c} - \frac{1}{2}\underset{\sim}{b}$

$\therefore \vec{BF} \cdot \vec{FE} = (\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b})(\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{b})$ ✓ dot product

$= \frac{1}{2}\underset{\sim}{a}\,\underset{\sim}{c} - \frac{1}{4}\underbrace{\cancel{\underset{\sim}{a}\underset{\sim}{b}}}_{0} - \underbrace{\cancel{\underset{\sim}{b}\underset{\sim}{c}}}_{0} + \frac{1}{2}\underset{\sim}{b}\,\underset{\sim}{b}$ ✓ result from (i)

$= \frac{1}{2}(\underset{\sim}{a} \cdot \underset{\sim}{c} + |\underset{\sim}{b}|^2)$

$= 0$ $\therefore \vec{BF} \perp \underset{\sim}{FE}$

(b) (i) Expand and simplify $3[(x-2)^3+2]$ 1

$$6$$

$$3\left[x^3 - 3(2)x^2 + 3(2)^2 x - 2^3 + 2\right]$$

$$= 3\left[x^3 - 6x^2 + 12x - 6\right]$$

$$= 3x^3 - 18x^2 + 36x - 18 \quad \boxed{\checkmark}$$

(ii) Given

$$I_n = \int_0^a \frac{x^n}{\sqrt{a^2-x^2}}\,dx\,,\ n \in \mathbb{N}$$

clearly show that:

$$I_n = \frac{a^2(n-1)}{n} I_{n-2}\quad,\ n \geq 2$$ 3

$$I_n = \int_0^a \underset{v}{x^{n-1}} \times \underset{u'}{x(a^2-x^2)^{-\frac{1}{2}}}\,dx$$

$\boxed{\checkmark \text{ integration by parts}}$

$$= \cancel{\left[\underset{v}{x^{n-1}} \times \left(-\underset{u}{(a^2-x^2)^{\frac{1}{2}}}\right)\right]_0^a} - \int_0^a \underset{v'}{(n-1)x^{n-2}}\left(-\underset{u}{(a^2-x^2)^{\frac{1}{2}}}\right)dx$$

$$+$$

$$= (n-1)\int_0^a x^{n-2}\sqrt{a^2-x^2}\,dx \times \frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}$$

$$= (n-1)\int_0^a \frac{x^{n-2}(a^2-x^2)}{\sqrt{a^2-x^2}}$$

$$= (n-1)\left[\int_0^a \frac{a^2x^{n-2}}{\sqrt{a^2-x^2}}\,dx - \int_0^a \frac{x^n}{\sqrt{a^2-x^2}}\,dx\right]$$

$$I_n = (n-1)\left[a^2 I_{n-2} - I_n\right]$$

$\boxed{\checkmark\ I_n \ \& \ I_{n-2}}$

$$\cancel{I_n} = a^2 n I_{n-2} - nI_n - a^2 I_{n-2} + \cancel{I_n}$$

$$nI_n = a^2 I_{n-2}(n-1)$$

$$\therefore I_n = \frac{a^2(n-1)}{n} I_{n-2}$$

$\boxed{\checkmark \text{ answer}}$

(iii) Show that

$$\int_2^4 \frac{3x^3 - 18x^2 + 36x - 18}{\sqrt{4x - x^2}}\,dx = 3I_3 + 3\pi$$

4

$$\text{LHS} = \int_2^4 \frac{3[(x-2)^3 + 2]}{\sqrt{4x - x^2}} \quad (\text{from (i)})$$

✓ substitution

$$= 3\int_2^4 \frac{(x-2)^3 + 2}{\sqrt{4 - (4 - 4x + x^2)}}\,dx$$

$$= 3\int_2^4 \frac{(x-2)^3 + 2}{\sqrt{4 - (x-2)^2}}\,dx$$

$u = x - 2$

$du = dx$

when $x = 4$, $u = 2$

when $x = 2$, $u = 0$

$$= 3\int_0^2 \frac{u^3 + 2}{\sqrt{2^2 - u^2}}\,du$$

$$= 3\int_0^2 \frac{u^3}{\sqrt{2^2 - u^2}} + 3\int_0^2 \frac{2}{\sqrt{2^2 - u^2}}$$

✓ complete square on denominator

$$= 3I_3 + 6\int_0^2 \frac{1}{\sqrt{2^2 - u^2}} \quad \left(\begin{matrix}\text{where}\\ a = 2\end{matrix}\right)$$

✓ use recurrence relation

$$= 3I_3 + 6\left[\sin^{-1}\left(\frac{u}{2}\right)\right]_0^2$$

$$= 3I_3 + \left[\sin^{-1}\left(\frac{2}{2}\right) - \cancel{\sin^{-1}}\left(\frac{0}{2}\right)\right]$$

✓ integrate to get $\sin^{-1}\frac{u}{2}$

$$= 3I_3 + 6\left(\frac{\pi}{2}\right)$$

$$= 3I_3 + 3\pi$$

$$= \text{RHS}$$

(iv) Hence or otherwise, evaluate

$$\int_2^4 \frac{3x^3 - 18x^2 + 36x - 18}{\sqrt{4x - x^2}} dx$$ **2**

$3I_3 + 3\pi$

$= \cancel{3}\left(\frac{2^2(3-1)}{\cancel{3}}\right)I_1 + 3\pi$

$= 8I_1 + 3\pi$

$= 8\int_0^2 \frac{x}{\sqrt{4-x^2}}dx + 3\pi$

$= 8\left[-\sqrt{4-x^2}\right]_0^2 + 3\pi$

$= 8\left[\cancel{-\sqrt{4-4}} - (-\sqrt{4-0^2})\right] + 3\pi$

$= 8(2) + 3\pi$

$= 16 + 3\pi$

$\boxed{\checkmark \text{ using reduction formula}}$

$\boxed{\checkmark \text{ correct answer}}$

Question 16 (15 Marks)

(a) (i) By considering the expansion of $k^3 - (k-1)^3$ show that:

$$\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1)$$ 2

*(do **NOT** use mathematical induction)*

$$k^3 - (k-1)^3 = \cancel{k^3} - (\cancel{k^3} - 3k^2 + 3k - 1)$$
$$= 3k^2 - 3k + 1$$

$$\therefore 3k^2 = k^3 - (k-1)^3 + 3k - 1$$

$$\therefore 3\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} \underbrace{k^3 - (k-1)^3}_{\text{telescoping}} + \sum_{k=1}^{n} \underbrace{3k-1}_{AP}$$

$$3\sum_{k=1}^{n} k^2 = \cancel{1^3} - 0^3 + \cancel{2^3} - \cancel{1^3} + \cancel{3^3} - \cancel{2^3} + \dots + n^3 + \cancel{(n-1)^3}$$
$$+ \frac{n}{2}[\underbrace{2(2)}_{2a} + \underbrace{(n-1)3}_{(n-1)d}]$$

$$\boxed{\checkmark \text{ telescoping series}}$$

$$= n^3 + \frac{n}{2}[4 + 3n - 3]$$
$$= n^3 + \frac{n}{2}[3n + 1]$$
$$= \frac{n}{2}[2n^2 + 3n + 1]$$

$$\boxed{\checkmark AP}$$

$$= \frac{n}{2}[2n^2 + 2n + n + 1]$$
$$= \frac{n}{2}[2n(n+1) + 1(n+1)]$$

$$3\sum_{k=1}^{n} k^2 = \frac{n}{2}(n+1)(2n+1)$$

$$\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1)$$

(ii) Prove that there are no finite sets of prime numbers where $2^2 + 3^2 + \cdots + n^2$ is a multiple of at least one element of the set for all integers $n \geq 2$. **2**

Assume that such a set exists, where the primes are numbered:

$p_1, p_2, p_3 \ldots p_n$

now, $2^2 + 3^2 + \ldots + n^2 = \frac{n}{6}(n+1)(2n+1) - 1$

(by rearranging part (i))

let $S = p_1 p_2 \ldots p_n$

& $n = 6S$

✓ substitution

$\therefore 2^2 + 3^2 + \ldots + n^2 = \frac{6S}{6}(S+1)(2S+1) - 1$

$= S(S+1)(2S+1) - 1$

as $S(S+1)(2S+1)$ is a multiple of all of the elements in the set

$S(S+1)(2S+1) - 1$ is not a multiple of any element & there is always an n such that $2^2 + 3^2 + \ldots + n^2$ is not a multiple of any element of the set

✓ explanation

(b) (i) Let $\underset{\sim}{w} = \underset{\sim}{u} + r\underset{\sim}{v}$ where $\underset{\sim}{w}$, $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are vectors and r is a scalar.

Prove that $|\underset{\sim}{v}|^2 r^2 + 2(\underset{\sim}{u} \cdot \underset{\sim}{v})r + |\underset{\sim}{u}|^2 \geq 0$ **1**

$|v^2|r^2 + 2(\underline{u}.\underline{v})r + |u|^2$

$= \underline{v}.\underline{v}\, r^2 + 2(\underline{u}.\underline{v})r + \underline{u}.\underline{u}$

$= (\underline{v}r + u).(\underline{v}r + \underline{u})$

$= \underline{w}.\underline{w}$

$= |w|^2$

≥ 0 $\boxed{\checkmark}$

(ii) Use your result from (i) to show that $\underset{\sim}{u} \cdot \underset{\sim}{v} \leq |\underset{\sim}{u}||\underset{\sim}{v}|$ **2**

as $\underbrace{|v|^2}_{a} r^2 + \underbrace{2(\underline{u}.\underline{v})}_{b} r + \underbrace{|u|^2}_{c} \geq 0$

$\Delta \leq 0$ $\boxed{\checkmark \text{ discriminant}}$

$\therefore [2(\underline{u}\ \underline{v})]^2 - 4|v|^2|u|^2 \leq 0$

$\therefore \cancel{4}(\underline{u}.\underline{v})^2 \leq \cancel{4}|u|^2|v|^2$

$(\underline{u}.\underline{v})^2 \leq (|u||v|)^2$

(as $|u||v| \geq 0$) $\therefore \underline{u}.\underline{v} \leq |u||v|$ $\boxed{\checkmark \text{ answer}}$

(iii) Use your result from part (ii) to show that for real numbers x, y and z.

let $xy + yz + zx \leq x^2 + y^2 + z^2$ **1**

$\underline{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\underline{v} = \begin{pmatrix} y \\ z \\ x \end{pmatrix}$

$xy + yz + zx \leq \left|\sqrt{x^2+y^2+z^2}\right|\left|\sqrt{x^2+y^2+z^2}\right|$

$\leq x^2 + y^2 + z^2$

(from ii) $\boxed{\checkmark}$

(iv) Use your result from part (iii) to show that for non-negative values of x, y and z:

$$xyz \leq \frac{x^3+y^3+z^3}{3}$$ 1

$$(xy+yz+zx)\underbrace{(x+y+z)}_{\times(x+y+z)} \leq (x^2+y^2+z^2)\underbrace{(x+y+z)}_{\times(x+y+z)}$$

$$\therefore \cancel{x^2y} + xyz + \cancel{zx^2} + \cancel{xy^2} + \cancel{y^2z} + zxy + zxy + \cancel{z^2y} + \cancel{z^2x}$$

$$\leq x^3 + \cancel{x^2y} + \cancel{x^2z} + \cancel{xy^2} + y^3 + \cancel{y^2z} + \cancel{z^2x} + \cancel{z^2y} + z^3$$

$$\therefore 3xyz \leq x^3+y^3+z^3$$

$$\therefore xyz \leq \frac{x^3+y^3+z^3}{3}$$ ✓

(v) Given that $a+b+c+d=0$ and $a^2+b^2+c^2+d^2=12$. Find: 2

1. The maximum value of $abcd$

as max value of abcd > 0
we can find max value
of |abcd| which is the same
as the max value of |a| |b| |c| |d|

✓ 2 applications of AM/GM inequality

$$\sqrt{|a||b||c||c|} = \frac{|a||b|}{2} + \frac{|c||d|}{2}$$

$$= \frac{\frac{a^2+b^2}{2}}{2} + \frac{\frac{c^2+d^2}{2}}{2}$$

$$= \frac{a^2+b^2+c^2+d^2}{4}$$

$$= \frac{12}{4}$$

$$= 3$$

3^2 ✓

$\therefore$ max value of $abcd = 9$

✓ answer

$a+b+c+d=0$ (given)

$\therefore \quad d=-(a+b+c)$

$\therefore \quad abcd=-abc(a+b+c)$

$\boxed{\checkmark \text{ rearrange to find max}}$

minimising $-abc(a+b+c)$ is the same as maximising $abc(a+b+c)$.

$a^2+b^2+c^2+d^2=12$ (given)

$\therefore \quad a^2+b^2+c^2+[-(a+b+c)]^2=12$ (sub in d)

$\therefore \quad a^2+b^2+c^2+\underline{a^2+b^2+c^2+2ab+2bc+2ca}=12$

$\therefore \quad 2(a^2+b^2+c^2+ab+bc+ca)=12$

$a^2+b^2+c^2+ab+bc+ca=6$ ①

$$\begin{aligned}(a+b+c)^2 &= a^2+b^2+c^2+2ab+2bc+2ca\\ &= a^2+b^2+c^2+ab+bc+ca+ab+bc+ca\\ &= 6+ab+bc+ca \quad \text{(from ①)}\\ &= 6+\frac{ab+bc+ca+ab+bc+ca}{2}\\ &\leq 6+\frac{a^2+b^2+c^2+ab+bc+ca}{2} \quad \text{(from iii)}\\ &= 6+\frac{6}{2} \quad \text{(from ①)}\\ &= 9\end{aligned}$$

$\boxed{\checkmark \text{ use (iii)}}$

$\therefore \quad (a+b+c)^2 \leq 9$ ②

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3}$$ (from (iv) where $x = \sqrt[3]{a}$, $y = \sqrt[3]{b}$, $z = \sqrt{c}$)

$$\therefore abc \leq \frac{(a+b+c)^3}{27}$$

$\boxed{\checkmark \text{ use (iv)}}$

$$abc(a+b+c) \leq \frac{(a+b+c)^4}{27}$$

$\times(a+b+c)$ $\qquad$ $\times (a+b+c)$

$$= \frac{[(a+b+c)^2]^2}{27}$$

(note that minimum value occurs with 3 positive & 1 negative so we take the case where $a, b, c \geq 0$)

$$\leq \frac{(9)^2}{27}$$

$$= \frac{81}{27}$$

$$= 3$$

as the maximum value of $abc(a+b+c)$ is 3 the minimum value of $-abc(a+b+c)$ must be -3

$\therefore$ minimum value of $abcd$ is -3

$\boxed{\checkmark \text{ answer}}$